1. domácí test z MAI 2 – řešení

(1) Na intervalu $(0,\pi)$ najděte primitivní funkci k funkci
$$f(x) = \frac{\sin^2 x}{3+\cos 2x}.$$

(i) fce $f(x) = \frac{\sin^2 x}{3+\cos 2x}$ je spojitá na $(0,\pi)$, tedy má na $(0,\pi)$ primitivní funkci

(ii) $f(x) = \frac{\sin^2 x}{3+\cos 2x} = \frac{\sin^2 x}{3+\cos^2 x-\sin^2 x}$ je fce sudá v $\sin x$ i $\cos x$, tedy lze pro výpočet primitivní funkce užít substituci $\operatorname{tg} x = t$

Pak: $$\int \frac{\sin^2 x}{3+\cos 2x}\,dx = \int \frac{\sin^2 x}{2+\cos^2 x+\cancel{\sin^2 x}+\cos^2 x-\cancel{\sin^2 x}}\,dx = \frac{1}{2}\int \frac{\sin^2 x}{1+\cos^2 x}\,dx$$

$\underset{\text{subst.}}{=}$ $\left|\begin{array}{l} \operatorname{tg} x = t \quad v\ (0,\frac{\pi}{2})\,! \\ x = \operatorname{arctg} t \\ x'(t) = \frac{1}{1+t^2} \quad a \quad \sin^2 x = \frac{t^2}{1+t^2} \\ \qquad\qquad\qquad\quad \cos^2 x = \frac{1}{1+t^2} \end{array}\right|$ $\underset{2VS}{=} \frac{1}{2}\int \frac{\frac{t^2}{1+t^2}}{\frac{1}{1+t^2}+1}\cdot\frac{1}{1+t^2}\,dt =$

$$= \frac{1}{2}\int \frac{t^2}{(1+t^2)(2+t^2)}\,dt \underset{\text{(rozklad)}}{=} \frac{1}{2}\left(\int \frac{2}{2+t^2}\,dt - \int\frac{1}{1+t^2}\,dt\right) =$$

$$= \frac{1}{2}\left(\int \frac{1}{1+\left(\frac{t}{\sqrt2}\right)^2}\,dt - \operatorname{arctg} t + c\right) = \frac{1}{\sqrt2}\operatorname{arctg}\left(\frac{t}{\sqrt2}\right) - \frac{1}{2}\operatorname{arctg} t + c$$

tedy, $\frac{1}{\sqrt2}\operatorname{arctg}\left(\frac{\operatorname{tg} x}{\sqrt2}\right) - \frac{1}{2}\operatorname{arctg}(\operatorname{tg} x) = \frac{1}{\sqrt2}\operatorname{arctg}\left(\frac{\operatorname{tg} x}{\sqrt2}\right) - \frac{1}{2}x + C \left(= F_0(x)+C\right)$
jsou ~~při~~ primit. fce k dané funkci v intervalu $(0,\frac{\pi}{2})$ (i v $(-\frac{\pi}{2},\frac{\pi}{2})$)
Díky π-periodicitě zadané funkce $f(x)$ je každá funkce $F_0(x)+C_1 = F_1(x)$ primitivní k $f(x)$ i v intervalu $(\frac{\pi}{2},\pi)$ (a tedy i „dále")
(Fce $F_0(x)$ je def. v každém intervalu $((2k-1)\frac{\pi}{2},(2k+1)\frac{\pi}{2})$, $k\in\mathbb{Z}$)

Zbývá tedy „napojit" primitivní funkci k dané $f(x)$ na „celém" intervalu $(0,\pi)$, kde k $f(x)$ primitivní funkce existuje –
– provedeme „slepení" primit. funkcí z intervalu $(0,\frac{\pi}{2})$ a $(\frac{\pi}{2},\pi)$ tak, aby primitivní funkce byla spojitá v bodě $x=\frac{\pi}{2}$, tj. aby
(tj. je třeba nalézt konstantu c_1 tak, aby)

$$\lim_{x\to\frac{\pi}{2}-} F_0(x) = \lim_{x\to\frac{\pi}{2}+} F_0(x) + c_1$$

zde:

$$\lim_{x\to\frac{\pi}{2}-}\left(\frac{1}{\sqrt{2}}\operatorname{arctg}\left(\frac{\operatorname{tg}x}{\sqrt{2}}\right)-\frac{1}{2}x\right) = \lim_{x\to\frac{\pi}{2}+}\left(\frac{1}{\sqrt{2}}\operatorname{arctg}\left(\frac{\operatorname{tg}x}{\sqrt{2}}\right)-\frac{1}{2}x+c_1\right)$$

tj.:

$$\frac{1}{\sqrt{2}}\cdot\frac{\pi}{2}-\cancel{\frac{\pi}{4}} = -\frac{1}{\sqrt{2}}\cdot\frac{\pi}{2}-\cancel{\frac{\pi}{4}}+c_1$$

$$c_1 = \sqrt{2}\cdot\frac{\pi}{2}$$

A tak: primitivní funkce k $f(x)$ jsou

$$F(x) = \frac{1}{\sqrt{2}}\operatorname{arctg}\left(\frac{\operatorname{tg}x}{\sqrt{2}}\right)-\frac{1}{2}x+c\ ,\quad x\in(0,\tfrac{\pi}{2}\rangle$$

$$F(x) = \frac{1}{\sqrt{2}}\left(\operatorname{arctg}\left(\frac{\operatorname{tg}x}{\sqrt{2}}\right)+\pi\right)-\frac{1}{2}x+c\ ,\quad x\in(\tfrac{\pi}{2},\pi)$$

nebo

(4) $$\int \frac{1}{x\sqrt{2+x-x^2}}\,dx = \underline{I}$$

(i) integrál existuje na intervalech $(-1,0)$ a $(0,2)$:

$\left(2+x-x^2 = (2-x)(x+1) > 0 \quad \text{a} \quad x \neq 0\right)$
v $(-1,0)$ a $(0,2)$

(ii) výpočet integrálu:

$$\int \frac{1}{x\sqrt{2+x-x^2}}\,dx = \int \frac{1}{x(x+1)\sqrt{\frac{2-x}{x+1}}}\,dx \quad (*)$$

$$\sqrt{2+x-x^2} = \sqrt{(2-x)(x+1)} = \sqrt{(x+1)^2\frac{2-x}{x+1}} = (x+1)\sqrt{\frac{2-x}{x+1}} \quad \left((x+1)>0\right.$$
$\left. \text{v } (-1,0)\cup(0,2)\right)$

Dále lze substituovat (2VS) $\sqrt{\frac{2-x}{x+1}} = t$, odkud $x = \frac{2-t^2}{1+t^2}$, $x'(t) = \frac{-6t}{(1+t^2)^2}$

a $(x+1) = \frac{3}{t^2+1}$

(ale je možné i zvolit „jinak“ na substituci $s = \sqrt{\frac{x+1}{2-x}}$, nebo Eulerovu substituci $\sqrt{2+x-x^2} = -\sqrt{2} + xt$)

Po substituci dostaneme (2VS)

$$\int \frac{1}{\frac{2-t^2}{1+t^2}\cdot\frac{3}{1+t^2}\cdot t}\cdot\left(\frac{-6t}{(1+t^2)^2}\right)dt = 2\int\frac{1}{t^2-2}\,dt = \int \frac{1}{\left(\frac{t}{\sqrt{2}}\right)^2-1}\,dt$$

a subst. $\frac{t}{\sqrt{2}} = u$ (2VS) $= \sqrt{2}\int\frac{1}{u^2-1}\,du = \frac{\sqrt{2}}{2}\int\left(\frac{1}{u-1}-\frac{1}{u+1}\right)du =$
$t' = \sqrt{2}$

(rozklad na parc. zlomky)

$$= \frac{\sqrt{2}}{2}\ln\left|\frac{u-1}{u+1}\right|$$, a tedy („zpět“)

$$\underline{I} = \frac{\sqrt{2}}{2}\ln\left|\frac{\frac{1}{\sqrt{2}}\sqrt{\frac{2-x}{1+x}}-1}{\frac{1}{\sqrt{2}}\sqrt{\frac{2-x}{1+x}}+1}\right| + C = \underline{\frac{\sqrt{2}}{2}\ln\left|\frac{\sqrt{2-x}-\sqrt{2}\sqrt{x+1}}{\sqrt{2-x}+\sqrt{2}\sqrt{x+1}}\right|} + C$$

(což lze „různě“ upravit, např. i dole)

2b (i) $\displaystyle\int_0^1 \frac{\arcsin\sqrt{x}}{\sqrt{1-x}}\,dx$ je integrál newtonův –

– fce $\dfrac{\arcsin\sqrt{x}}{\sqrt{1-x}}$ je spojitá na $\langle 0,1)$, ale neomezená v $\langle 0,1)$ $\left(\lim\limits_{x\to 1-}\dfrac{\arcsin\sqrt{x}}{\sqrt{1-x}}=+\infty\right)$, tedy není integrál Riemannův

(ii) výpočet: (například)

$$\int_0^1 \frac{\arcsin\sqrt{x}}{\sqrt{1-x}}\,dx \underset{2VS}{=} \left|\begin{array}{l} \sqrt{x}=t \\ x=t^2 \\ x'=2t \\ x=0\to t=0 \\ x=1\to t=1 \end{array}\right| = 2\int_0^1 \arcsin t\cdot\frac{t}{\sqrt{1-t^2}}\,dt =$$

$$\underset{\text{per partes}}{=} \left|\begin{array}{ll} u'=\dfrac{t}{\sqrt{1-t^2}} \quad \left(\dfrac{-2t}{\sqrt{1-t^2}} \text{ a 1VS}\right) & u=-\sqrt{1-t^2} \\ v=\arcsin t & v'=\dfrac{1}{\sqrt{1-t^2}} \end{array}\right| =$$

$$= 2\left(\underbrace{\left[-\sqrt{1-t^2}\cdot\arcsin t\right]_0^1}_{=0} + \int_0^1 1\,dt\right) = \underline{\underline{2}}$$

nebo „přímo" per partes (asi i jednodušší)

$$\int_0^1 \frac{\arcsin\sqrt{x}}{\sqrt{1-x}}\,dx \underset{pp}{=} \left|\begin{array}{ll} u'=\dfrac{1}{\sqrt{1-x}}, & u=-2\sqrt{1-x} \\ v=\arcsin\sqrt{x}, & v'=\dfrac{1}{\sqrt{1-x}}\cdot\dfrac{1}{2\sqrt{x}} \end{array}\right| =$$

$$= \underbrace{\left[-2\sqrt{1-x}\cdot\arcsin\sqrt{x}\right]_0^1}_{=0} + \int_0^1 \frac{2\cancel{\sqrt{1-x}}}{\cancel{\sqrt{1-x}}\cdot 2\sqrt{x}}\,dx = 2\left[\sqrt{x}\right]_0^1 = \underline{\underline{2}}$$

③ Obsah omezené rovinné oblasti ω, která je ohraničená osou x, přímkou $x = a, a > 0$ a grafem fce $f(x) = \ln(x + \sqrt{1+x^2})$.

$S(\omega) = \int_0^a \ln(x+\sqrt{1+x^2})\,dx$ (fce $\ln(x+\sqrt{1+x^2}) = f(x)$ je rostoucí v R
$\left(\left(\ln(x+\sqrt{1+x^2})\right)' = \frac{1}{\sqrt{1+x^2}},\ f(0)=0 \right)$)

a výpočet: (nepř.)

$$\int_0^a \ln(x+\sqrt{1+x^2})\,dx \underset{PP}{=} \left| \begin{array}{ll} u'=1 & u=x \\ v=\ln(x+\sqrt{1+x^2}), & v'=\frac{1}{\sqrt{1+x^2}} \end{array} \right|$$

$$= \left[x\ln(x+\sqrt{1+x^2})\right]_0^a - \int_0^a \frac{x}{\sqrt{1+x^2}}\,dx \underset{(1VS\ -\ (1+x^2)'=2x)}{=}$$

$$= a\ln(a+\sqrt{1+a^2}) - \left[\sqrt{1+x^2}\right]_0^a =$$

$$= a\ln(a+\sqrt{1+a^2}) - \sqrt{1+a^2} + 1$$

„Pomalý" výpočet:

$$\int_0^a \frac{x}{\sqrt{1+x^2}}\,dx = \int_0^a \frac{2x}{2\sqrt{1+x^2}} = \left| \begin{array}{l} 1+x^2 = t \\ 2x = t' \\ x=0 \to t=1 \\ x=a \to t=1+a^2 \end{array} \right| = \int_1^{1+a^2} \frac{1}{2\sqrt{t}}\,dt$$

$$= \left[\sqrt{t}\right]_1^{1+a^2} = \sqrt{1+a^2} - 1$$

(nebo (primit. fce) $\int \frac{2x}{2\sqrt{1+x^2}}\,dx \underset{1VS}{=} \int \frac{dt}{2\sqrt{t}} = \sqrt{t} + c = \sqrt{1+x^2} + c$)

(4) Objem rotačního tělesa, které vznikne rotací oblasti ω kolem osy x, je-li

$$\omega = \{[x,y] \in \mathbb{R}^2;\ 0 \le x \le 1 \ \wedge\ 0 \le y \le \sqrt{x}\,\mathrm{arctg}\,x\}$$

$$V(\Omega) = \pi \int_0^1 \left(\sqrt{x}\,\mathrm{arctg}\,x\right)^2 dx = \pi \int_0^1 x\,\mathrm{arctg}^2 x\,dx =$$

$$\underset{\text{per partes}}{=} \left| \begin{array}{ll} u' = x & u = \frac{x^2}{2} \\ v = \mathrm{arctg}^2 x, & v' = 2\,\mathrm{arctg}\,x \cdot \frac{1}{1+x^2} \end{array} \right| =$$

$$= \pi \left\{ \left[\frac{x^2}{2}\mathrm{arctg}^2 x\right]_0^1 - \int_0^1 \frac{x^2}{1+x^2}\,\mathrm{arctg}\,x\,dx \right\} = \left(\frac{x^2}{1+x^2} = \frac{x^2+1}{x^2+1} - \frac{1}{1+x^2} \right)$$

$$= \pi \left\{ \frac{1}{2}\left(\frac{\pi}{4}\right)^2 - \left(\int_0^1 \mathrm{arctg}\,x\,dx - \int_0^1 \frac{\mathrm{arctg}\,x}{1+x^2}\,dx \right) \right\} =$$

$$= \pi \left\{ \frac{\pi^2}{32} - \left[x\,\mathrm{arctg}\,x - \frac{\ln(1+x^2)}{2}\right]_0^1 + \left[\frac{\mathrm{arctg}^2 x}{2}\right]_0^1 \right\} =$$

$$= \pi \left(\frac{\pi^2}{32} + \frac{\pi^2}{32} - \frac{\pi}{4} + \frac{1}{2}\ln 2 \right) = \underline{\underline{\pi \left(\frac{\pi^2}{16} - \frac{\pi}{4} + \frac{1}{2}\ln 2 \right)}}$$

Použité výpočty:

1)
$$\int \mathrm{arctg}\,x\,dx \underset{\text{PP}}{=} \left| \begin{array}{ll} u' = 1 & u = x \\ v = \mathrm{arctg}\,x, & v' = \frac{1}{1+x^2} \end{array} \right| = x\,\mathrm{arctg}\,x - \int \frac{x}{1+x^2}\,dx$$

$$\underset{1VS}{=} x\,\mathrm{arctg}\,x - \frac{1}{2}\ln(1+x^2) + C$$

$$\left(\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C \right)$$

$$(f'(x) \text{ spojitá},\ f(x) \neq 0 \text{ v } J)$$

5) Délka grafu funkce $f(x) = \arcsin x + \sqrt{1-x^2}$, $x \in [-1,1]$

$$l = \int_{-1}^{1} \sqrt{1+f'^2(x)}\,dx$$

$$(f'(x))^2 = \left(\left(\arcsin x + \sqrt{1-x^2}\right)'\right)^2 = \left(\frac{1}{\sqrt{1-x^2}} + \frac{-\cancel{2}x}{\cancel{2}\sqrt{1-x^2}}\right)^2 =$$

$$= \frac{(1-x)^2}{1-x^2} = \frac{1-x}{1+x}$$

fce $\sqrt{1+f'^2(x)} = \sqrt{1+\frac{1-x}{1+x}} = \sqrt{\frac{2}{1+x}}$ – spojitá v $(-1,1\rangle$, ale není omezená v $(-1,1\rangle$! Tedy integrál musíme počítat jako Newtonův – zůstane zachován „smysl"? Tj. „délka grafu"?

Výpočet l:

$$l = \int_{-1}^{1} \sqrt{1+f'^2(x)}\,dx = \int_{-1}^{1} \sqrt{\frac{2}{x+1}}\,dx = \sqrt{2}\int_{-1}^{1} \frac{1}{\sqrt{x+1}}\,dx =$$

$$= 2\sqrt{2}\left[\sqrt{x+1}\right]_{-1}^{1} = 2\sqrt{2}\cdot\sqrt{2} = \underline{\underline{4}}$$

(existuje jako Newtonův)